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ON SOME PROBLEMS

IN THE

CALCULUS OF FINITE DIFFERENCES.

BY

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INTRODUCTORY.

The idea of writing the following paper was suggested to the author while conducting the Second Examination of the Faculty of Actuaries in Scotland, by the evident difficulty which many of the students experienced in finding a text book on the subject of Finite Differences which exactly suited their requirements. The elementary treatises on algebra barely pass the threshold of the subject, while those works on the calculus itself embrace a much wider range than the examinations require, and assume a greater amount of knowledge on the part of the student than the elementary works usually afford him.

As indicated by its title, this essay does not profess to be by any means a complete treatise on the application of the calculus of Finite Differences, but merely seeks, in some measure, to aid the student to graduate from the elementary works on the one hand to the higher treatises on the other.

The paper was read before the Actuarial Society of Edinburgh towards the end of 1871, and having since been somewhat enlarged and modified, it is now printed by the Society, chiefly in the hope that it may be found helpful to those who are prosecuting their actuarial studies.

Edinburgh,
16*th November*, 1876.

ON SOME PROBLEMS IN THE CALCULUS OF FINITE DIFFERENCES.

THE problems which form the subject of this paper arise from series having their general term of the form

$$c_0v^0+c_1v^1+c_2v^2+c_3v^3+\dots\dots+c_nv^n.$$

In this expression, c_0, c_1, c_2, &c., are constant quantities which may have any values, positive or negative, integral or fractional, including 0; and v is the variable, the powers of which must be positive and integral.

The series consists of an infinite number of terms, from that in which $v=0$ onwards, v increasing by unity each term; and backwards, v decreasing by unity each term.

The terms of the series are denoted by u with a suffix which indicates the value of the variable in that particular term, thus u_3 denotes the term in which v has the value 3 in the expression for the general term.

When the general term of a series is known, that is, when values are given to the constants, the terms of the series are formed by substituting 0 as the value of v in u_0, 1 as its value in u_1, 2 as its value in u_2, and so on; thus, if the general term of a series be $1+4v-3v^2+v^3$, the terms of the series are found as follows,

$$\begin{array}{lllll}
u_{-2}=1 & +4\times(-2) & -3\times4 & +(-8) & =-27\\
u_{-1}=1 & +4\times(-1) & -3\times1 & +(-1) & =-\ 7\\
u_0\ =1 & +4\times\ \ 0 & -3\times0 & +\ \ 0 & =\ \ \ 1\\
u_1\ =1 & +4\times\ \ 1 & -3\times1 & +\ \ 1 & =\ \ \ 3\\
u_2\ =1 & +4\times\ \ 2 & -3\times4 & +\ \ 8 & =\ \ \ 5\\
u_3\ =1 & +4\times\ \ 3 & -3\times9 & +\ 27 & =\ \ 13
\end{array}$$

When the highest power of the variable involved in the general term is the nth, the series is said to be of the nth degree. Arithmetical progressions are series of this kind, of the first degree.

The various problems connected with such series are solved by the method of differences, and deal with the relations subsisting between the terms, differences, and sums of the series.

The terms of the 1st, 2nd, 3rd, nth orders of differences are denoted by Δ^1, Δ^2, Δ^3, Δ^n respectively, and to show to which term they apply, its symbol is written immediately after them; thus the term of the first order of differences corresponding to the term of the series u_x is denoted by $\Delta^1 u_x$, and the term of the nth order of differences corresponding to the term u_{x+p} is denoted by $\Delta^n u_{x+p}$.

On Diagram I. is shown the way in which the differences are formed from the terms of a series commencing with u_x.

If the terms of the series are written underneath each other, as in the column headed u, the terms of the 1st order of differences are formed from them by subtracting each term of the series from the succeeding one, and writing the remainders in the next column (headed $\Delta^1 u$) opposite the respective terms which were subtracted. So that the first difference of u_x is $u_{x+1}-u_x=\Delta^1 u_x$; on this equation of definition the Calculus of Finite Differences is based.

The terms of the 2nd order of differences are formed from those of the first order in precisely the same way as the latter were formed from the terms of the series, so that $\Delta^2 u_x=\Delta^1 u_{x+1}-\Delta^1 u_x$; and giving these their values, we have $\Delta^2 u_x=(u_{x+2}-u_{x+1})-(u_{x+1}-u_x)=u_{x+2}-2u_{x+1}+u_x$.

In a similar way each succeeding order of differences is formed from the one before it. This is carried on in the diagram to the 4th order, and referring to it we observe that the coefficients of the terms of the series in each term of the 1st order of differences are $1-1$, of the 2nd order are $1-2+1$, of the 3rd order are $1-3+3-1$, and of the 4th order are $1-4+6-4+1$. These coefficients are the same as those of the powers of x in the expansion of the binomial $(x-1)$ to the 1st, 2nd, 3rd, and 4th powers respectively; from the foregoing we might expect that a law similar to that of the Binomial Theorem will be found to govern the formation of the terms of the several orders of differences, the symbols x^n, x^{n-1}, &c., in the expansion of $(x-1)^n$ corresponding to u_{x+n}, u_{x+n-1}, &c., respectively in the expression for $\Delta^n u_x$, and we give expression to this hypothesis in the formula

$$\Delta^n u_x=u_{x+n}-\frac{n}{1}u_{x+n-1}+\frac{n(n-1)}{1.2}u_{x+n-2}-\text{\&c.}$$

To prove that this formula is true for all positive and integral values of n, assume it to hold for the nth order of differences, then we have

$$\begin{array}{rcl}
\Delta^n u_{x+1} & = & u_{x+n+1} - \frac{n}{1} u_{x+n} + \frac{n(n-1)}{1.2} u_{x+n-1} - \frac{n(n-1)(n-2)}{1.2.3} u_{x+n-2} + \&c. \\
-\Delta^n u_x & = & - u_{x+n} + \frac{n}{1} u_{x+n-1} - \frac{n(n-1)}{1.2} u_{x+n-2} + \&c. \\
\hline
\Delta^n u_{x+1} - \Delta^n u_x & = & u_{x+n+1} - \frac{(n+1)}{1} u_{x+n} + \frac{(n+1)n}{1.2} u_{x+n-1} - \frac{(n+1)n(n-1)}{1.2.3} u_{x+n-2} + \&c.
\end{array}$$

and $\Delta^n u_{x+1} - \Delta^n u_x = \Delta^{n+1} u_x$, therefore

$$\Delta^{n+1} u_x = u_{x+n+1} - \frac{(n+1)}{1} u_{x+n} + \frac{(n+1)n}{1.2} u_{x+n-1} - \frac{(n+1)n(n-1)}{1.2.3} u_{x+n-2} + \&c.$$

an expression the same as that given above for $\Delta^n u_x$, except that in this case $n+1$ is substituted for n, so that if the formula is true for the nth order of differences it is true for the $(n+1)$th order; but from the diagram it is seen to hold up to the 4th order, and since it holds for the 4th it must hold for the 5th, and if for the 5th then for the 6th, and so for all higher orders.

In the kind of series we are considering, the orders of differences terminate after a certain definite number, and we shall now consider in what way they are limited.

When the series is of the nth degree—that is, when the highest power of the variable in the general term is v^n—the terms of the nth order of differences have all the same value, or the nth order of differences is said to be constant, and consequently the terms of all the higher orders equal 0.

This may be proved as follows,

Let $\quad u_v = av^n + a_1 v^{n-1} + a_2 v^{n-2} + a_3 v^{n-3} + \&c.,$

which is a rational integral function of v of the nth degree, then

$${}_v = u_{v+1} - u_v = \{a(v+1)^n + a_1(v+1)^{n-1} + a_2(v+1)^{n-2} + \&c.\} - \{av^n + a_1 v^{n-1} + a_2 v^{n-2} + \&c$$

$$= \left\{ \begin{array}{l}
u_{v+1} = \left\{ \begin{array}{lcllll}
a(v+1)^n & = & av^n & + a\frac{n}{1} v^{n-1} & + a\frac{n(n-1)}{1.2} v^{n-2} & + a\frac{n(n-1)(n-2)}{1.2.3} v^{n-3} + \& \\
+a_1(v+1)^{n-1} & = & & + a_1 v^{n-1} & + a_1 \frac{n-1}{1} v^{n-2} & + a_1 \frac{(n-1)(n-2)}{1.2} v^{n-3} + \& \\
+a_2(v+1)^{n-2} & = & & & + a_2 v^{n-2} & + a_2 \frac{n-2}{1} v^{n-3} + \& \\
+a_3(v+1)^{n-3} & = & & & & + a_3 v^{n-3} + \& \\
+ \quad \&c. & = & & & & + \&
\end{array} \right. \\
-u_v \qquad\qquad\qquad = -av^n - a_1 v^{n-1} - a_2 v^{n-2} - a_3 v^{n-3} - \&
\end{array} \right.$$

$${}_v \qquad = anv^{n-1} + b_1 v^{n-2} + b_2 v^{n-3} + \&$$

where b_1, b_2, &c., are constant coefficients, being composed only of the constant quantities n, a, a_1, a_2, &c.

Thus we find that Δu_v is a rational integral function of v, of a degree one lower than the original function—viz., of the $(n-1)$th degree; and the like operation of differencing may therefore be performed on it. Thus $\Delta^2 u_v = \{an(v+1)^{n-1} + b_1(v+1)^{n-2} + b_2(v+1)^{n-3} + \&c.\} - \{anv^{n-1} + b_1 v^{n-2} + b_2 v^{n-3} + \ \&c.\}$, which we find in a similar manner to that shown above,

$$= an(n-1)v^{n-2} + c_1 v^{n-3} + c_2 v^{n-4} + \ \&c.$$

where c_1, c_2, &c., are also constant coefficients. So that $\Delta^2 u_v$ is a rational integral function of v of the degree $(n-2)$. Carrying on this process in a similar manner we at last find that

$$\Delta^n u_v = a.n(n-1)(n-2) \ldots . \ 3.2.1$$

a constant quantity.

An example will show this more distinctly; let us therefore take $u_v = v^3 - 3v^2 + 4v + 1$, then,

$$\begin{array}{lllr}
\Delta u_v = & \left\{ \begin{array}{l} \\ \\ u_{v+1} \quad = \\ \\ \\ -u_v \end{array} \right. & \left\{ \begin{array}{l} (v+1)^3 = \\ -3(v+1)^2 = \\ +4(v+1)^1 = \\ +1 \qquad\quad = \end{array} \right. & \begin{array}{r} v^3+3v^2+3v+1 \\ -3v^2-6v-3 \\ +4v+4 \\ +1 \end{array} \\
 & & \qquad\qquad\quad = & -v^3+3v^2-4v-1 \\ \hline
\Delta^1 u_v & & \qquad\qquad\quad = & 3v^2-3v+2 \\ \hline \hline
\Delta^2 u_v = & \left\{ \begin{array}{l} \Delta^1 u_{v+1} = \\ \\ \\ -\Delta^1 u_v \end{array} \right. & \left\{ \begin{array}{l} 3(v+1)^2 = \\ -3(v+1) \ = \\ +2 \qquad\quad = \end{array} \right. & \begin{array}{r} 3v^2+6v+3 \\ -3v-3 \\ +2 \end{array} \\
 & & \qquad\qquad\quad = & -3v^2+3v-2 \\ \hline
\Delta^2 u_v & & \qquad\qquad\quad = & 6v+0 \\ \hline \hline
\Delta^3 u_v = & \left\{ \begin{array}{l} \Delta^2 u_{v+1} = \\ -\Delta^2 u_v \end{array} \right. & \begin{array}{l} 6(v+1) = \\ \qquad\qquad = \end{array} & \begin{array}{r} 6v+6 \\ -6v \end{array} \\ \hline
\Delta^3 u_v & & \qquad\qquad\quad = & +6 = 3.2.1 \\ \hline \hline
\end{array}$$

a constant quantity, as found above.

We shall now consider the law of formation of the terms of the series when one term and its differences are given.

On Diagram II., the series is supposed to have its 4th order of differences constant, and the term given is u_x with its differences, $\Delta^1 u_x$, $\Delta^2 u_x$, $\Delta^3 u_x$, &c. Now since the 4th order of differences is constant, and since $\Delta^4 u_x$ is known, all the terms of the 4th order of differences are known, and we may use c as the symbol of the value of each term.

Then as $\Delta^n u_x = \Delta^{n-1} u_{x+1} - \Delta^{n-1} u_x$, therefore $\Delta^{n-1} u_{x+1} = \Delta^{n-1} u_x + \Delta^n u_x$; and as $\Delta^3 u_x$ and the terms of the 4th order of differences are known, we can form the other terms of the 3rd order of differences onwards by continued additions,

$$\text{thus}\quad \begin{aligned} \Delta^3 u_{x+1} &= \Delta^3 u_x + \Delta^4 u_x = && = \Delta^3 u_x + c \\ \Delta^3 u_{x+2} &= \Delta^3 u_{x+1} + \Delta^4 u_{x+1} = (\Delta^3 u_x + c) && + c = \Delta^3 u_x + 2c \\ \Delta^3 u_{x+3} &= \Delta^3 u_{x+2} + \Delta^4 u_{x+2} = (\Delta^3 u_x + 2c) && + c = \Delta^3 u_x + 3c \\ \&\text{c.} &= && \&\text{c.} \end{aligned}$$

and also since $\Delta^n u_{x-1} = \Delta^{n-1} u_x - \Delta^{n-1} u_{x-1}$, and consequently $\Delta^{n-1} u_{x-1} = \Delta^{n-1} u_x - \Delta^n u_{x-1}$, we can form the terms of the 3rd order of differences backwards from $\Delta^3 u_x$ by continual subtraction,

$$\text{thus}\quad \begin{aligned} \Delta^3 u_{x-1} &= \Delta^3 u_x - \Delta^4 u_{x-1} && = \Delta^3 u_x - c \\ \Delta^3 u_{x-2} &= \Delta^3 u_{x-1} - \Delta^4 u_{x-2} = (\Delta^3 u_x - c) && - c = \Delta^3 u_x - 2c \\ \Delta^3 u_{x-3} &= \Delta^3 u_{x-2} - \Delta^4 u_{x-3} = (\Delta^3 u_x - 2c) && - c = \Delta^3 u_x - 3c \\ \&\text{c.} &= && \&\text{c.} \end{aligned}$$

In a similar manner each of the preceding orders of differences is formed from the difference of u_x of that order, and the next higher order of differences, backwards, till we come to the terms of the series.

On examining the expressions for the terms of the series, we find that the coefficients of u_x and its differences in the expression for u_{x+1} are $1+1$, in that for u_{x+2} they are $1+2+1$, in that for u_{x+3} they are $1+3+3+1$, &c. These are the same as the coefficients of the powers of x in the expansion of the binomial $(1+x)$ to the 1st, 2nd, 3rd, &c., powers respectively, $\Delta^1 u_x$ corresponding to x^1, $\Delta^2 u_x$ to x^2, $\Delta^3 u_x$ to x^3, &c. From this correspondence we assume that a law similar to that which governs the formation of the coefficients in the expansions of the binomial also governs the formation of the coefficients in the expressions for the terms of the series, and this we express by the formula

$$u_{x+n} = u_x + \frac{n}{1}\Delta^1 u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \frac{n(n-1)(n-2)}{1.2.3}\Delta^3 u_x + \&\text{c.}$$

This formula may be proved to hold true for all values of n, whether positive or negative, integral or fractional, by a similar method to that employed above in proving the formula for the value of $\Delta^n u_x$ expressed in the terms of the series.

In the expression Δu_x, u_x is the symbol of a quantity, viz., the term of a series, while Δ is the symbol of an operation to be performed on it, viz., the subtraction of it from u_{x+1}; from which operation results a new quantity, represented by the combined symbols of operation and quantity.

The symbol of operation, Δ, has a definite signification only when combined with symbols of quantity, but it may be separated, and operations performed on it alone similar to those which may be performed on symbols of quantity, it being afterwards united with the symbols of quantity.

Since $u_{x+1}=u_x+\Delta u_x$, separating symbols of operation from those of quantity, we have $(1+\Delta)u_x=u_{x+1}$, the symbol $(1+\Delta)$ signifying that to a certain quantity (in this case represented by u_x) is to be added its difference. Again, $u_{x+2}=u_{x+1}+\Delta u_{x+1}=(1+\Delta)u_{x+1}$, and substituting the value of u_{x+1} we have $u_{x+2}=(1+\Delta)(1+\Delta)u_x=(1+\Delta)^2u_x$; here the symbol $(1+\Delta)^2$ signifies, first, that to the quantity represented by u_x should be added its difference, and, second, that to the resulting quantity, viz., $u_x+\Delta u_x$, or u_{x+1}, its difference should also be added; and similarly $u_{x+n}=(1+\Delta)^n u_x$.

Expanding the symbol $(1+\Delta)^n$ by the Binomial Theorem, we have

$$(1+\Delta)^n=1+\frac{n}{1}\Delta^1+\frac{n(n-1)}{1.2}\Delta^2+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3+\ \&c.;$$

and now replacing the symbol of quantity

$$(1+\Delta)^n u_x,\ \text{or}\ u_{x+n}=u_x+\frac{n}{1}\Delta^1 u_x+\frac{n(n-1)}{1.2}\Delta^2 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3 u_x+\ \&c.$$

the same expression as we obtained above.

We thus see how closely the mode of formation of the expression for u_{x+n} in terms of u_x and its differences corresponds with the expression for $(1+x)^n$ as expanded by the Binomial Theorem.

In a similar manner, to find the formula for the value of $\Delta^n u_x$ expressed in the terms of the series, write $\Delta^n u_x$ in the form $\{(1+\Delta)-1\}^n u_x$, and, separating the symbol of quantity, develope the first part of the expression, by which we have

$$\{(1+\Delta)-1\}^n=(1+\Delta)^n-\frac{n}{1}(1+\Delta)^{n-1}+\frac{n(n-1)}{1.2}(1+\Delta)^{n-2}-\&c.$$

again replacing the symbol of quantity,

$$\{(1+\Delta)-1\}^n u_x=(1+\Delta)^n u_x-\frac{n}{1}(1+\Delta)^{n-1}u_x+\frac{n(n-1)}{1.2}(1+\Delta)^{n-2}u_x-\ \&c.$$

and giving these their values, as explained above,

$$\Delta^n u^x=u_{x+n}-\frac{n}{1}u_{x+n-1}+\frac{n(n-1)}{1.2}u_{x+n-2}-\ \&c.,$$

the same expression as that formerly obtained.

The method of dealing with the symbols of operation, separate from those of quantity, will be found useful in determining the form of expression for many formulæ in the Calculus of Differences.

In dealing with the sums of series which we have now to consider, the following symbols are made use of in addition to those already defined.

Σu_{x+n}, or its equivalent $\Delta^{-1}u_{x+n}$, denotes the sum of all the terms of the series preceding u_{x+n}, and it represents that function of the variable the first difference of which is u_{x+n}, so that $\Delta(\Sigma u_{x+n})=\Sigma u_{x+n+1}-\Sigma u_{x+n}=u_{x+n}$, or Δ and Σ cancel each other. Σu_x, Σu_{x+1}, &c., being each the sum of an infinite number of terms, have no definite values by themselves; but we have only to deal with the differences of these sums, such as $\Sigma u_{x+n}-\Sigma u_x$, and each of these differences being the sum of a finite number of terms has a definitely ascertainable value.

$S_n u_x$ denotes the sum of n terms of a series, from u_x to u_{x+n-1} both inclusive, and is equal to $\Sigma u_{x+n}-\Sigma u_x$, or to the sum of all the terms of the series up to u_{x+n-1} inclusive, less their sum to u_{x-1} inclusive.

The sum of the first n terms of that part of the series with which we are dealing may also be denoted simply by S_n.

$S_n(v^p)$ denotes the sum of the natural numbers from 1 to n, each raised to the pth power, viz., $1^p+2^p+3^p+\ldots\ldots n^p$.

Referring to Diagram II., the terms of the column headed Σu are formed from those of Column u in exactly the same way as the latter are formed from the terms of Column $\Delta^1 u$, and consequently the expression for Σu_{x+n} corresponds with that for u_{x+n}, viz.,

$$\Sigma u_{x+n}=\Sigma u_x+\frac{n}{1}u_x+\frac{n(n-1)}{1.2}\Delta^1 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_x+\text{\&c.}$$

and as $$S_n u_x=\Sigma u_{x+n}-\Sigma u_x,$$

therefore $$S_n u_x=\frac{n}{1}u_x+\frac{n(n-1)}{1.2}\Delta^1 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_x+\text{\&c.}$$

Again, since it was shown above that $u_{x+n}=(1+\Delta)^n u_x$, and that $\Sigma u_{x+n}=\Delta^{-1}u_{x+n}$, therefore $\Sigma u_{x+n}=\Delta^{-1}(1+\Delta)^n u_x$, which may be put into the equivalent form $(1+\Delta)^n\Delta^{-1}u_x$, then separating $\Delta^{-1}u_x$ and expanding $(1+\Delta)^n$ we have

$$(1+\Delta)^n=1+\frac{n}{1}\Delta^1+\frac{n(n-1)}{1.2}\Delta^2+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3+\text{\&c.};$$

replacing the symbol of quantity,

$$(1+\Delta)^n(\Delta^{-1}u_x)=(\Delta^{-1}u_x)+\frac{n}{1}\Delta^1(\Delta^{-1}u_x)+\frac{n(n-1)}{1.2}\Delta^2(\Delta^{-1}u_x)$$
$$+\frac{n(n-1)(n-2)}{1.2.3}\Delta^3(\Delta^{-1}u_x)+\text{\&c.}$$

giving these their equivalent values.

$$\Sigma u_{x+n}=\Sigma u_x+\frac{n}{1}u_x+\frac{n(n-1)}{1.2}\Delta^1 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_x+\text{\&c.}$$

and

$$S_n u_x=\Sigma u_{x+n}-\Sigma u_x=\frac{n}{1}u_x+\frac{n(n-1)}{1.2}\Delta^1 u_x+\frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_x+\text{\&c.}$$

The formula for $S_n u_x$ may be found in yet another way, by summing separately the coefficients of u_x, $\Delta^1 u_x$, $\Delta^2 u_x$, &c., in the expressions for the terms of the series as given in Column u of Diagram II. Doing this we find that $S_n u_x = (u_x$ into the first n terms of the 1st order of figurate numbers, viz., $1+1+1+1+$ &c. $= \frac{n}{1} u_x) + (\Delta^1 u_x$ into the first $n-1$ terms of the 2nd order of figurate numbers, viz., $1+2+3+4+$ &c. $= \frac{(n-1)n}{1.2} \Delta^1 u_x) + (\Delta^2 u_x$ into the first $n-2$ terms of the 3rd order, viz., $1+3+6+10+$&c. $= \frac{(n-2)(n-1)n}{1.2.3} \Delta^2 u_x) + (\Delta^3 u_x$ into the first $n-3$ terms of the 4th order, viz., $1+4+10+20+$ &c. $= \frac{(n-3)(n-2)(n-1)n}{1.2.3.4} \Delta^3 u_x)$ $+$ &c.; and writing out these values we have as before

$$S_n u_x = \frac{n}{1} u_x + \frac{n(n-1)}{1.2} \Delta^1 u_x + \frac{n(n-1)(n-2)}{1.2.3} \Delta^2 u_x \\ + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \Delta^3 u_x + \text{&c.}$$

By a similar method to the above we may find the expression for the sum of n terms of a series beginning with u_1, in terms of n and the constants of the general term.

From the formula $u_v = c_0 v^0 + c_1 v^1 + c_2 v^2 +$ &c., we have

$$\begin{array}{l} u_1 = 1c_0 + 1c_1 + 1c_2 + 1c_3 + 1c_4 + \text{&c.} \\ u_2 = 1,, +2,, + 4,, + 8,, + 16,, + \text{&c.} \\ u_3 = 1,, +3,, + 9,, + 27,, + 81,, + \text{&c.} \\ u_4 = 1,, +4,, +16,, + 64,, +256,, + \text{&c.} \\ u_5 = 1,, +5,, +25,, +125,, +625,, + \text{&c.} \\ \vdots \\ u_n = 1,, +n,, +n^2,, + n^3,, + n^4,, + \text{&c.} \end{array}$$

and, summing the columns perpendicularly, we find $S_n(u_v) =$ $S_n(v^0) \times c_0 + S_n(v^1) \times c_1 + S_n(v^2) \times c_2 + S_n(v^3) \times c_3 + S_n(v^4) \times c_4 +$ &c., and by finding the expressions for the sums of the 0, 1, 2, 3, 4, &c., powers of the natural numbers each to n terms and substituting them in the above expression we find

$$S_n(u_v) = n.c_0 + \frac{1}{2} n(n+1).c_1 + \frac{1}{6} n(n+1)(2n+1).c_2 + \frac{1}{4} n^2(n+1)^2.c_3 \\ + \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1).c_4 + \text{&c.}$$

EXAMPLES AND DEDUCTIONS.

The formulæ employed will be found on page 26, numbered as they are referred to in the examples.

(I.) Find the differences of u_5 when $u_5=55$, $u_6=126$, $u_7=259$, $u_8=484$, $u_9=837$, and Δ^4 is constant.

This is done by forming a table thus,

u	$\Delta^1 u$	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$
$u_5=55$	71	62	30	6
$u_6=126$	133	92	36	
$u_7=259$	225	128		
$u_8=484$	353			
$u_9=837$				

and the differences of u_5 are $\Delta^1 u_5=71$, $\Delta^2 u_5=62$, $\Delta^3 u_5=30$, and $\Delta^4 u_5=6$.

(II.) Find $\Delta^4 u_2$ when $u_2=201$, $u_3=303$, $u_4=443$, $u_5=630$, and $u_6=874$.

By formula No. 3,

$$\begin{aligned}\Delta^4 u_2 &= u_{2+4}-\frac{4}{1}u_{2+3}+\frac{4\cdot 3}{1.2}u_{2+2}-\frac{4.3.2}{1.2.3}u_{2+1}+\frac{4.3.2.1}{1.2.3.4}u_2\\ &=u_6-4u_5+6u_4-4u_3+u_2\\ &=874-2{,}520+2{,}658-1{,}212+201\\ &=3{,}733-3{,}732=1.\end{aligned}$$

(III.) Find u_{12} when $u_5=55$, $u_6=126$, $u_7=259$, $u_8=484$, $u_9=837$, and Δ^4 is constant.

As in the first example, we find the differences of u_5 are $\Delta^1 u_5=71$, $\Delta^2 u_5=62$, $\Delta^3 u_5=30$, $\Delta^4 u_5=6$. Then by formula No. 2(a),

$$\begin{aligned}u_{12}=u_{5+7}&=u_5+7\Delta^1 u_5+21\Delta^2 u_5+35\Delta^3 u_5+35\Delta^4 u_5\\ &=55+497+1{,}302+1{,}050+210\\ &=3{,}114.\end{aligned}$$

(IV.) In the formula for u_{x+n} used above, n is positive, but it will equally apply when n is negative, of which we shall now take an example, viz., to find u_2 from the data given in the last example, By formula No. 2(*b*) when *n* is negative,

$$u_2=u_{5-3}=u_5+\frac{(-3)}{1}\Delta^1u_5+\frac{(-3)(-4)}{1.2}\Delta^2u_5$$
$$+\frac{(-3)(-4)(-5)}{1.2.3}\Delta^3u_5+\frac{(-3)(-4)(-5)(-6)}{1.2.3.4}\Delta^4u_5$$
$$=u_5-3\Delta^1u_5+6\Delta^2u_5-10\Delta^3u_5+15\Delta^4u_5,$$

and substituting the values of u_5 and its differences as in the last example,

$$u_{5-3}=55-213+372-300+90$$
$$=517-513=4.$$

(V.) Find the sum of the first 16 terms of the series 1, 3, 5, 13, 33, &c.

Find the differences of u_1, as below,

u	Δ^1u	Δ^2u	Δ^3u	Δ^4u
$u_1=1$	2	0	6	0
$u_2=3$	2	6	6	
$u_3=5$	8	12		
$u_4=13$	20			
$u_5=33$				

By formula No. 5,

$$S_{16}=16u_1+120\Delta^1u_1+560\Delta^2u_1+1{,}820\Delta^3u_1$$
$$=16+240+0+10{,}920$$
$$=11{,}176.$$

(VI.) Find u_7 when $u_3=1{,}001, u_4=1{,}024, u_5=1{,}060, u_6=1{,}118, u_8=1{,}376, u_9=1{,}649$, and Δ^5 is constant.

By formula No. 3,

$$\Delta^6u_3=u_9-6u_8+15u_7-20u_6+15u_5-6u_4+u_3,$$

but $\Delta^6=0$ since Δ^5 is constant, therefore

$$15u_7=-u_9+6u_8+20u_6-15u_5+6u_4-u_3$$
$$=-1{,}649+8{,}256+22{,}360-15{,}900+6{,}144-1{,}001$$
$$=36{,}760-18{,}550$$
$$=18{,}210 \qquad \text{and } u_7=1{,}214.$$

(VII.) Find the series commencing with u_1, of which, $u_5=89$, $u_7=265$, $u_9=593$, and $u_{11}=1{,}121$, Δ^3 being constant.

Since Δ^3 is constant, we obtain (from formula No. 2(*a*) and the values of the terms given) four simultaneous equations, and from these we find by continuous subtraction the values of u_1 and its differences, from which we form the series, thus

$$\begin{array}{llllllll}
u_5=u_{1+4}=1u_1+ & 4\Delta^1u_1+ & 6\Delta^2u_1+ & 4\Delta^3u_1= & 89 & \quad(1)\\
u_7=u_{1+6}=1\text{ ,, }+ & 6\text{ ,, } & +15\text{ ,, } & +\ 20\text{ ,, } = & 265 & \\
u_9=u_{1+8}=1\text{ ,, }+ & 8\text{ ,, } & +28\text{ ,, } & +\ 56\text{ ,, } = & 593 & \\
u_{11}=u_{1+10}=1\text{ ,, }+ & 10\text{ ,, } & +45\text{ ,, } & +120\text{ ,, } = & 1{,}121 & \\
\hline
\end{array}$$

$$\begin{array}{lllll}
2\Delta^1u_1+ & 9\Delta^2u_1+ & 16\Delta^3u_1= & 176 & \quad(2)\\
2\text{ ,, } & +13\text{ ,, } & +36\text{ ,, } = & 328 & \\
2\text{ ,, } & +17\text{ ,, } & +64\text{ ,, } = & 528 & \\
\hline
\end{array}$$

$$\begin{array}{llll}
4\Delta^2u_1+ & 20\Delta^3u_1= & 152 & \quad(3)\\
4\text{ ,, } & +28\text{ ,, } = & 200 & \\
\hline
\end{array}$$

$$8\Delta^3u_1=48 \qquad (4)$$

From equation

(4) we have $8\Delta^3u_1=48$ and $\Delta^3u_1=6$

(3) ,, $4\Delta^2u_1=152-20\Delta^3u_1=152-120=32$ and $\Delta^2u_1=8$

(2) ,, $2\Delta^1u_1=176-16\Delta^3u_1-9\Delta^2u_1=176-96-72=8$ and $\Delta^1u_1=4$

(1) ,, $1u_1=89-4\Delta^3u_1-6\Delta^2u_1-4\Delta^1u_1$
$=89-24-48-16=1$ and $u_1=1$

Having thus got u_1 and its differences, we form the series

u	Δ^1u	Δ^2u	Δ^3u
$u_1=1$	4	8	6
$u_2=5$	12	14	6
$u_3=17$	26	20	
$u_4=43$	46		
$u_5=89$			
&c.			

(VIII.) Find u_{13} when $u_2=5$, $u_5=71$, $u_7=225$, $u_{11}=1013$, and Δ^3 is constant.

The solution of this might be obtained in a similar way to that of the preceding example, but it may also be found by employing Lagrange's formula for interpolation, which is, that if there are given certain values of u_v, as u_a, u_b, u_c, u_d, &c., corresponding to values of the variable, a, b, c, d, &c., then

$$u_v=\frac{(v-b)(v-c)(v-d)\text{ \&c.}}{(a-b)(a-c)(a-d)\text{ \&c.}}u_a+\frac{(v-a)(v-c)(v-d)\text{ \&c.}}{(b-a)(b-c)(b-d)\text{ \&c.}}u_b$$
$$+\frac{(v-a)(v-b)(v-d)\text{ \&c.}}{(c-a)(c-b)(c-d)\text{ \&c.}}u_c+\text{\&c.}$$

By this formula

$$u_{13}=\frac{(13-5)(13-7)(13-11)}{(2-5)(2-7)(2-11)}u_2+\frac{(13-2)(13-7)(13-11)}{(5-2)(5-7)(5-11)}u_5$$

$$+\frac{(13-2)(13-5)(13-11)}{(7-2)(7-5)(7-11)}u_7+\frac{(13-2)(13-5)(13-7)}{(11-2)(11-5)(11-7)}u_{11}$$

$$=-\frac{8.6.2}{3.5.9}\cdot 5+\frac{11.6.2}{3.2.6}\cdot 71-\frac{11.8.2}{5.2.4}\cdot 225+\frac{11.8.6}{9.6.4}\cdot 1{,}013$$

$$=-\frac{32}{9}+\frac{781}{3}-990+\frac{22{,}286}{9}$$

$$=\frac{1}{9}(-32+2{,}343-8{,}910+22{,}286)=\frac{1}{9}(24{,}629-8{,}942)=\frac{1}{9}(15{,}687)$$

$$=1{,}743$$

(IX.) If instead of finding the value of a specified term, as u_{13}, it be desired to find the analytical expression for u_v in a series of ascending powers of v, then using the data given in the foregoing example, we have

$$u_v=\frac{(v-5)(v-7)(v-11)}{(2-5)(2-7)(2-11)}u_2+\frac{(v-2)(v-7)(v-11)}{(5-2)(5-7)(5-11)}u_5$$

$$+\frac{(v-2)(v-5)(v-11)}{(7-2)(7-5)(7-11)}u_7+\frac{(v-2)(v-5)(v-7)}{(11-2)(11-5)(11-7)}u_{11}$$

$$=-\frac{1}{27}(v^3-23v^2+167v-385)+\frac{71}{36}(v^3-20v^2+113v-154)$$

$$-\frac{45}{8}(v^3-18v^2+87v-110)+\frac{1{,}013}{216}(v^3-14v^2+59v-70)$$

$$=\frac{1}{216}\left\{\begin{array}{rrrr} -\quad 8v^3 & +\quad 184v^2 & -\quad 1{,}336v & +\quad 3{,}080 \\ +\quad 426\,,, & -\quad 8{,}520\,,, & +\quad 48{,}138\,,, & -\quad 65{,}604 \\ -1{,}215\,,, & +21{,}870\,,, & -105{,}705\,,, & +133{,}650 \\ +1{,}013\,,, & +14{,}182\,,, & +\quad 59{,}767\,,, & -\quad 70{,}910 \\ \hline \end{array}\right.$$

$$=\frac{1}{216}\left\{\begin{array}{rrrr} +1{,}439v^3 & +22{,}054v^2 & +107{,}905v^1 & +136{,}730 \\ -1{,}223\,,, & -22{,}702\,,, & -107{,}041\,,, & -136{,}514 \\ \hline \end{array}\right.$$

$$=\frac{1}{216}\Big\}+\ 216v^3\ -\ 648v^2\ +\ 864v\ +\ 216$$

$$=v^3-3v^2+4v+1.$$

(X.) Given $u_0=1$, $u_5=71$, $u_{10}=741$, and $u_{15}=2{,}761$, Δ^3 being constant, interpolate 4 terms between each of these terms, or find the series u_0, u_1, u_2, u_3, &c.

The series of equidistant terms may be represented by $u_{0(5)}$, $u_{1(5)}$, $u_{2(5)}$, $u_{3(5)}$, &c., so that $\Delta^1 u_{0(5)}=u_{1(5)}-u_{0(5)}$, and the series of interpolated terms may be represented by $u_{0(1)}$, $u_{1(1)}$, $u_{2(1)}$, $u_{3(1)}$, &c. The differences of $u_{0(5)}$ are found as in Example (I.), thus

	u	$\Delta^1 u$	$\Delta^2 u$	$\Delta^3 u$
$u_{0(5)}=$	1	70	600	750
$u_{1(5)}=$	71	670	1350	
$u_{2(5)}=$	741	2020		
$u_{3(5)}=$	2761			

so that $\Delta^1 u_{0(5)}=70$, $\Delta^2 u_{0(5)}=600$, and $\Delta^3 u_{0(5)}=750$.

Then by formula No. 2(*c*) giving n the successive fractional values, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, we have,

$$u_{1(1)}=u_{0(5)+\frac{1}{5}(5)}=1u_{0(5)}+\frac{\frac{1}{5}}{1}\Delta^1 u_{0(5)}+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{1.2}\Delta^2 u_{0(5)}+\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)}{1.2.3}\Delta^3 u_{0(5)}$$

$$u_{2(1)}=u_{0(5)+\frac{2}{5}(5)}=1\ \text{,,}\ +\frac{\frac{2}{5}}{1}\ \text{,,}\ +\frac{\frac{2}{5}\left(\frac{2}{5}-1\right)}{1.2}\ \text{,,}\ +\frac{\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)}{1.2.3}\ \text{,,}$$

$$u_{3(1)}=u_{0(5)+\frac{3}{5}(5)}=1\ \text{,,}\ +\frac{\frac{3}{5}}{1}\ \text{,,}\ +\frac{\frac{3}{5}\left(\frac{3}{5}-1\right)}{1.2}\ \text{,,}\ +\frac{\frac{3}{5}\left(\frac{3}{5}-1\right)\left(\frac{3}{5}-2\right)}{1.2.3}\ \text{,,}$$

reducing the fractions, and substituting the values of $u_{0(5)}$ and its differences, as found above,

$$u_{1(1)}=1+\frac{1}{5}(70)-\frac{2}{25}(600)+\frac{6}{125}(750)=1+14-48+36=51-48=\ 3$$

$$u_{2(1)}=1+\frac{2}{5}(70)-\frac{3}{25}(600)+\frac{8}{125}(750)=1+28-72+48=77-72=\ 5$$

$$u_{3(1)}=1+\frac{3}{5}(70)-\frac{3}{25}(600)+\frac{7}{125}(750)=1+42-72+42=85-72=13$$

Having now got the values of $u_{0(1)}$, $u_{1(1)}$, $u_{2(1)}$, and $u_{3(1)}$, we can find the differences of $u_{0(1)}$ (as the same order of differences is constant in the interpolated series as in the series of equidistant terms), and carry on the series as far as desired,—thus

	u	$\Delta^1 u$	$\Delta^2 u$	$\Delta^3 u$
$u_{0(1)}=$	1	2	0	6
$u_{1(1)}=$	3	2	6	6
$u_{2(1)}=$	5	8	12	6
$u_{3(1)}=$	13	20	18	
$u_{4(1)}=$	33	38		
$u_{5(1)}=$	71			

This example might also have been solved by the method employed in Example VII. It is on this problem that the second method of interpolation referred to at p. 64 of Mr. Meikle's "*Observations on the Rate of Mortality of Assured Lives, as experienced by Ten Assurance Companies in Scotland,*" is founded; and it is by an adaptation of it that the mortality tables of the Institute of Actuaries were graduated by Mr. Woolhouse, as explained by him in the Introduction to the "*Institute of Actuaries Life Tables,*" p. lxxxvii, et seq.

(XI.) Find the expression for the differences of $u_{0(1)}$ in the series $(u_0+u_1+u_2+$ &c.) or the equivalent series $(u_{0(1)}+u_{1(1)}+u_{2(1)}+$ &c.), in terms of the differences of $u_{0(b)}$ in the series $(u_0+u_b+u_{2b}+$ &c.), or the equivalent series $(u_{0(b)}+u_{1(b)}+u_{2(b)}+$ &c.), when the third order of differences is assumed to be constant.

The terms of the series $u_{0(b)}+u_{1(b)}+$ &c. are equidistant terms of the series $u_{0(1)}+u_{1(1)}+$ &c., so that $u_{n(b)}=u_{nb(1)}$, or the general terms of the two series may be considered the same, except that the variable, in the series $u_{0(1)}+u_{1(1)}+$ &c., increases by unity each term while in the series $u_{0(b)}+u_{1(b)}+$ &c., it increases by b each term.

Let Δ be the symbol of differences applicable to the series $u_{0(1)}+u_{1(1)}+$ &c., so that $\Delta u_{0(1)}=u_{1(1)}-u_{0(1)}$, and let δ be the symbol of differences applicable to the series $u_{0(b)}+u_{1(b)}+$ &c., so that $\delta u_{0(b)}=u_{1(b)}-u_{0(b)}$.

By formula No. 2 (c) we have

$$
\begin{array}{llllllll}
u_{0(1)}=u_{0(b)+\frac{0}{b}(b)}=1u_{0(b)} & & & & & & & \\
u_{1(1)}=u_{0(b)+\frac{1}{b}(b)}=1 \;,, & +\dfrac{\frac{1}{b}}{1}\,\delta^1 u_{0(b)} & + & \dfrac{\frac{1}{b}(\frac{1}{b}-1)}{1.2} & \delta^2 u_{0(b)} & + & \dfrac{\frac{1}{b}(\frac{1}{b}-1)(\frac{1}{b}-2)}{1.2.3} & \delta^3 u_{0(b)} \\
u_{2(1)}=u_{0(b)+\frac{2}{b}(b)}=1 \;,, & +\dfrac{\frac{2}{b}}{1}\;\;,, & + & \dfrac{\frac{2}{b}(\frac{2}{b}-1)}{1.2} & ,, & + & \dfrac{\frac{2}{b}(\frac{2}{b}-1)(\frac{2}{b}-2)}{1.2.3} & ,, \\
u_{3(1)}=u_{0(b)+\frac{3}{b}(b)}=1 \;,, & +\dfrac{\frac{3}{b}}{1}\;\;,, & + & \dfrac{\frac{3}{b}(\frac{3}{b}-1)}{1.2} & ,, & + & \dfrac{\frac{3}{b}(\frac{3}{b}-1)(\frac{3}{b}-2)}{1.2.3} & ,,
\end{array}
$$

reducing the fractions, we have the four linear equations immediately following; we then proceed by continuous subtraction, thus,

$$\begin{array}{llllllll}
u_{0(1)}= & 1u_{0(b)} & +0\,\frac{\delta^1u_{0(b)}}{1.b} & +\;0 & \frac{\delta^2u_{0(b)}}{1.2.b^2} & + & 0\;. & \frac{\delta^3u_{0(b)}}{1.2.3b^3}\\
u_{1(1)}= & 1\;,, & +1\;,, & +(1-b) & ,, & + & (1-3b+2b^2) & ,,\\
u_{2(1)}= & 1\;,, & +2\;,, & +(4-2b) & ,, & + & (8-12b+4b^2) & ,,\\
u_{3(1)}= & 1\;,, & +3\;,, & +(9-3b) & ,, & + & (27-27b+6b^2) & ,,\\
\hline
\Delta^1u_{0(1)}= & & 1\frac{\delta^1u_{0(b)}}{1.b} & +(1-b) & \frac{\delta^2u_{0(b)}}{1.2.b^2} & + & (1-3b+2b^2) & \frac{\delta^3u_{0(b)}}{1.2.3b^3}.\\
\Delta^1u_{1(1)}= & & 1\;,, & +(3-b) & ,, & + & (7-9b+2b^2) & ,,.\\
\Delta^1u_{2(1)}= & & 1\;,, & +(5-b) & ,, & + & (19-15b+2b^2) & ,,\\
\hline
\Delta^2u_{0(1)}= & & & 2 & \frac{\delta^2u_{0(b)}}{1.2.b^2} & + & (6-6b) & \frac{\delta^3u_{0(b)}}{1.2.3b^3}\\
\Delta^2u_{1(1)}= & & & 2 & ,, & + & (12-6b) & ,,\\
\hline
\Delta^3u_{0(1)}= & & & & & + & 6 & \frac{\delta^3u_{0(b)}}{1.2.3b^3}
\end{array}$$

We have thus found the expressions for the differences of $u_{0(1)}$ in terms of the differences of $u_{0(b)}$, and these may be put in the form

$$\Delta^1u_{0(1)}=\frac{\delta^1u_{0(b)}}{b}+\frac{1-b}{2}\,\frac{\delta^2u_{0(b)}}{b^2}+\frac{(1-b)(1-2b)}{2.3}\,\frac{\delta^3u_{0(b)}}{b^3}$$

$$\Delta^2u_{0(1)}=\frac{\delta^2u_{0(b)}}{b^2}+(1-b)\,\frac{\delta^3u_{0(b)}}{b_3}$$

$$\Delta^3u_{0(1)}=\frac{\delta^3u_{0(b)}}{b^3}$$

(XII.) Find a general expression for the difference of $u_{0(a)}$, of any order, in the series $u_{0(a)}+u_{1(a)}+u_{2(a)}+$ &c., in terms of the differences of $u_{0(b)}$ in the series $u_{0(b)}+u_{1(b)}+u_{2(b)}+$ &c.

The general term of the series $u_{0(a)}+a_{1(a)}+$ &c., or $u_{v(a)}$, being the same as the general term of the series $u_{0(b)}+u_{1(b)}+$ &c., except that the variable in the first series increases by a each term, and is denoted by $v_{(a)}$; and the variable in the second series increases by b, and is denoted by $v_{(b)}$; therefore $u_{nb(a)}=u_{na(b)}$.

Let Δ be the symbol of differences applicable to the series $u_{0(a)}+u_{1(a)}+$ &c., and δ that applicable to $u_{0(b)}+u_{1(b)}+$ &c.

Since $v_{(a)}$ increases by a each term, its increment $\Delta v_{(a)}=a$; and since $v_{(b)}$ increases by b each term, its increment $\delta v_{(b)}=b$; therefore $\delta v_{(b)}=\frac{a}{b}\Delta v_{(a)}$.

Also since $u_{1(a)}=u_{\frac{a}{b}(b)}$, therefore $(1+\Delta)u_{0(a)}=(1+\delta)^{\frac{a}{b}}u_{0(b)}$, separating symbols of operation from those of quantity.

And expanding this equation,

$$(1+\Delta)u_{0(a)}=\left\{1+\frac{\frac{a}{b}}{1}\delta+\frac{\frac{a}{b}\left(\frac{a}{b}-1\right)}{1.2}\delta^2+\frac{\frac{a}{b}\left(\frac{a}{b}-1\right)\left(\frac{a}{b}-2\right)}{1.2.3}\delta^3+\text{\&c.}\right\}u_{0(b)},$$

replacing the symbols of quantity,

$$u_{0(a)}+\Delta u_{0(a)}=u_{0(b)}+\frac{a}{1}\cdot\frac{\delta^1 u_{0(b)}}{b}+\frac{a(a-b)}{1.2}\cdot\frac{\delta^2 u_{0(b)}}{b^2}+\frac{a(a-b)(a-2b)}{1.2.3}\cdot\frac{\delta^3 u_{0(b)}}{b^3}+\text{\&c.}$$

Now, since $u_{0(a)}=u_{0(b)}$, subtracting these equals and substituting $\Delta v_{(a)}$ for a, and $\delta v_{(b)}$ for b,

$$\Delta u_{0(a)}=\frac{\Delta v_{(a)}}{1}\cdot\frac{\delta u_{0(b)}}{\delta v_{(b)}}+\frac{\Delta v_{(a)}(\Delta v_{(a)}-\delta v_{(b)})}{1.2}\cdot\frac{\delta^2 u_{0(b)}}{(\delta v_{(b)})^2}$$
$$+\frac{\Delta v_{(a)}(\Delta v_{(a)}-\delta v_{(b)})(\Delta v_{(a)}-2\delta v_{(b)})}{1.2.3}\cdot\frac{\delta^3 u_{0(b)}}{(\delta v_{(b)})^3}+\text{\&c.},$$

dividing each side by $\Delta v_{(a)}$,

$$\frac{\Delta u_{0(a)}}{\Delta v_{(a)}}=\frac{\delta u_{0(b)}}{\delta v_{(b)}}+\frac{(\Delta v_{(a)}-\delta v_{(b)})}{1.2}\cdot\frac{\delta^2 u_{0(b)}}{(\delta v_{(b)})^2}+\frac{(\Delta v_{(a)}-\delta v_{(b)})(\Delta v_{(a)}-2\delta v_{(b)})}{2.3}\cdot\frac{\delta^3 u_{0(b)}}{(\delta v_{(b)})^3}+\text{\&}$$

and generally,

$$\frac{\Delta^n u_{0(a)}}{(\Delta v_{(a)})^n}=\left\{\frac{\delta u_{0(b)}}{\delta v_{(b)}}+\frac{(\Delta v_{(a)}-\delta v_{(b)})}{2}\cdot\frac{\delta^2 u_{0(b)}}{(\delta v_{(b)})^2}+\frac{(\Delta v_{(a)}-\delta v_{(b)})(\Delta v_{(a)}-2\delta v_{(b)})}{2.3}\cdot\frac{\delta^3 u_{0(b)}{}^3}{(\delta v_{(b)})^3}+\text{\&c.}\right\}$$

(XIII.) Find the series of which $S_5u_1=1365$, $S_{10}u_1=5155$, $S_{15}u_1=13370$, and $S_{20}u_1=28635$, Δ^3 being constant.

Since Δ^3 is constant, we obtain from formula No. 5, and the values given, 4 linear equations, from which we find the values of u_1 and its differences, by continuous subtraction, as follows,

$$\begin{array}{lrrrrl}
S_5u_1= & 5u_1+ & 10\Delta^1u_1+ & 10\Delta^2u_1+ & 5\Delta^3u_1= & 1,365 \\
S_{10}u_1= & 10\text{,,}+ & 45\text{ ,, }+ & 120\text{ ,, }+ & 210\text{ ,, }= & 5,155 \\
S_{15}u_1= & 15\text{,,}+ & 105\text{ ,, }+ & 455\text{ ,, }+ & 1,365\text{ ,, }= & 13,370 \\
S_{20}u_1= & 20\text{,,}+ & 190\text{ ,, }+ & 1,140\text{ ,, }+ & 4,845\text{ ,, }= & 28,635 \\
\hline
 & 5u_1+ & 10\Delta^1u_1+ & 10\Delta^2u_1+ & 5\Delta^3u_1= & 1,365\quad(1) \\
 & 5\text{,,}+ & 35\text{ ,, }+ & 110\text{ ,, }+ & 205\text{ ,, }= & 3,790 \\
 & 5\text{,,}+ & 60\text{ ,, }+ & 335\text{ ,, }+ & 1,155\text{ ,, }= & 8,215 \\
 & 5\text{,,}+ & 85\text{ ,, }+ & 685\text{ ,, }+ & 3,480\text{ ,, }= & 15,265 \\
\hline
 & & 25\Delta^1u_1+ & 100\Delta^2u_1+ & 200\Delta^3u_1= & 2,425\quad(2) \\
 & & 25\text{ ,, }+ & 225\text{ ,, }+ & 950\text{ ,, }= & 4,425 \\
 & & 25\text{ ,, }+ & 350\text{ ,, }+ & 2,325\text{ ,, }= & 7,050 \\
\hline
 & & & 125\Delta^2u_1+ & 750\Delta^3u_1= & 2,000\quad(3) \\
 & & & 125\text{ ,, }+ & 1,375\text{ ,, }= & 2,625 \\
\hline
 & & & & 625\Delta^3u_1= & 625\quad(4)
\end{array}$$

We obtain from equation

$$625\Delta^3 u_1 = 625 \qquad \text{and } \Delta^3 u_1 = 1$$
$$125\Delta^2 u_1 = 2{,}000 - 750\Delta^3 u_1 = 2{,}000 - 750 = 1{,}250 \qquad \Delta^2 u_1 = 10$$
$$25\Delta^1 u_1 = 2{,}425 - 200\Delta^3 u_1 - 100\Delta^2 u_1 = 2{,}425 - 200 - 1{,}000 = 1{,}225 \quad \Delta^1 u_1 = 49$$
$$5u_1 = 1{,}365 - 5\Delta^3 u_1 - 10\Delta^2 u_1 - 10\Delta^1 u_1 = 1{,}365 - 5 - 100 - 490 = 770 \qquad u_1 = 154$$

Having now got the values of u_1 and its differences, we can form the series thus

u	$\Delta^1 u$	$\Delta^2 u$	$\Delta^3 u$
$u_1 = 154$	49	10	1
$u_2 = 203$	59	11	1
$u_3 = 262$	70	12	
$u_4 = 332$	82		
$u_5 = 414$			
&c.			

It is on this problem that the first method of interpolation explained by Mr. Meikle, at p. 58 of his *Observations on Mortality,* before referred to, is founded.

(XIV.) The foregoing example may also be solved in a similar way to Example (X).

Since $S_5 u_1 = \Sigma u_6 - \Sigma u_1$, $S_{10} u_1 = \Sigma u_{11} - \Sigma u_1$, &c., assuming $\Sigma u_1 = Q$, we have $\Sigma u_6 = 1365 + Q$, $\Sigma u_{11} = 5155 + Q$, &c., these are equidifferent terms of the series Σu_1, Σu_2, Σu_3, &c., distant from each other by 5 terms, and we may therefore denote Σu_1 by $U_{0(5)}$, Σu_6 by $U_{1(5)}$, &c.; so that $\Delta U_{0(5)} = \Sigma u_6 - \Sigma u_1$, and $U_{0(5)+\frac{n}{5}(5)} = \Sigma u_{1+n}$. Also since the series u_1, u_2, u_3, &c., had Δ^3 constant, the series of sums, or $U_{0(5)}$, $U_{1(5)}$, &c., will have Δ^4 constant.

We find the differences of $U_{0(5)}$ as follows,

	U	Δ^1U	Δ^2U	Δ^3U	Δ^4U
$U_{0(5)} = \Sigma u_1 = Q =$	Q	1,365	2,425	2,000	625
$U_{1(5)} = \Sigma u_6 = S_5 u_1 + Q =$	Q + 1,365	3,790	4,425	2,625	
$U_{2(5)} = \Sigma u_{11} = S_{10} u_1 + Q =$	Q + 5,155	8,215	7,050		
$U_{3(5)} = \Sigma u_{16} = S_{15} u_1 + Q =$	Q + 13,370	15,265			
$U_{4(5)} = \Sigma u_{21} = S_{20} u_1 + Q =$	Q + 28,635				

and from them we have

$$U_{0(5)} = Q, \quad \frac{\Delta^1 U_{0(5)}}{5} = 273, \quad \frac{\Delta^2 U_{0(5)}}{25} = 97, \quad \frac{\Delta^3 U_{0(5)}}{125} = 16, \quad \frac{\Delta^4 U_{0(5)}}{625} = 1;$$

then by formula No. 2(c) (or No. 5, with the substitutions explained above), giving n the fractional values $\frac{0}{5}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{5}$, we obtain the following equations,

$$
\begin{array}{llllll}
U_{0(5)+\frac{0}{5}(5)}=1U_{0(5)} & +\frac{0}{5}\Delta^1U_{0(5)} & +\quad 0\quad \Delta^2U_{0(5)} & +\quad 0\quad \Delta^3U_{0(5)} & +\quad 0 & \Delta^4U \\
U_{0(5)+\frac{1}{5}(5)}=1\ ,, & +\frac{1}{5}\ ,, & +\frac{\frac{1}{5}(\frac{1}{5}-1)}{1.2}\ ,, & +\frac{\frac{1}{5}(\frac{1}{5}-1)(\frac{1}{5}-2)}{1.2.3}\ ,, & +\frac{\frac{1}{5}(\frac{1}{5}-1)(\frac{1}{5}-2)(\frac{1}{5}-3)}{1.2.3.4} & \\
U_{0+(5)\frac{2}{5}(5)}=1\ ,, & +\frac{2}{5}\ ,, & +\frac{\frac{2}{5}(\frac{2}{5}-1)}{1.2}\ ,, & +\frac{\frac{5}{2}(\frac{2}{5}-1)(\frac{2}{5}-2)}{1.2.3}\ ,, & +\frac{\frac{2}{5}(\frac{2}{5}-1)(\frac{2}{5}-2)(\frac{2}{5}-3)}{1.2.3.4} & \\
U_{0(5)+\frac{3}{5}(5)}=1\ ,, & +\frac{3}{5}\ ,, & +\frac{\frac{3}{5}(\frac{3}{5}-1)}{1.2}\ ,, & +\frac{\frac{3}{5}(\frac{3}{5}-1)(\frac{3}{5}-2)}{1.2.3}\ ,, & +\frac{\frac{3}{5}(\frac{3}{5}-1)(\frac{3}{5}-2)(\frac{3}{5}-3)}{1.2.3.4} & \\
U_{0(5)+\frac{4}{5}(5)}=1\ ,, & +\frac{4}{5}\ ,, & +\frac{\frac{4}{5}(\frac{4}{5}-1)}{1.2}\ ,, & +\frac{\frac{4}{5}(\frac{4}{5}-1)(\frac{4}{5}-2)}{1.2.3}\ ,, & +\frac{\frac{4}{5}(\frac{4}{5}-1)(\frac{4}{5}-2)(\frac{4}{5}-3)}{1.2.3.4} & \\
U_{0(5)+\frac{5}{5}(5)}=1\ ,, & +\frac{5}{5}\ ,, & +\frac{\frac{5}{5}(\frac{5}{5}-1)}{1.2}\ ,, & +\quad 0\quad ,, & +\quad 0 & \\
\hline
\end{array}
$$

$$
\begin{array}{llllllllll}
=1U_{0(5)} & +0\frac{\Delta^1U_{0(5)}}{5} & & -0\frac{\Delta^2U_{0(5)}}{25} & & +0\frac{\Delta^3U_{0(5)}}{125} & & -\ 0\frac{\Delta^4U_{0(5)}}{625} & \\
=1 & +1 & ,, & -2 & ,, & +6 & ,, & -21 & ,, \\
=1 & +2 & ,, & -3 & ,, & +8 & ,, & -26 & ,, \\
=1 & +3 & ,, & -3 & ,, & +7 & ,, & -21 & ,, \\
=1 & +4 & ,, & -2 & ,, & +4 & ,, & -11 & ,, \\
=1 & +5 & ,, & -0 & ,, & +0 & ,, & -\ 0 & ,, \\
\hline
\end{array}
$$

$$
\begin{array}{llll}
=Q & =Q & =\Sigma u_1 & \text{as shown above.} \\
=Q+\ 273-194+\ 96-21 & =Q+\ 154 & =\Sigma u_2 & ,, \\
=Q+\ 546-291+128-26 & =Q+\ 357 & =\Sigma u_3 & ,, \\
=Q+\ 819-291+112-21 & =Q+\ 619 & =\Sigma u_4 & ,, \\
=Q+1{,}092-194+\ 64-11 & =Q+\ 951 & -\Sigma u_5 & ,, \\
=Q+1{,}365 & =Q+1{,}365 & =\Sigma u_6 & ,, \\
\end{array}
$$

Having thus obtained the six consecutive sums of the terms, by differencing we obtain the terms of the series and their differences, from which we carry on the series in the usual way, thus,

Σ	u	$\Delta^1 u$	$\Delta^2 u$	$\Delta^3 u$
$\Sigma u_1=Q$	$u_1=154$	49	10	1
$\Sigma u_2=Q+\ 154$	$u_2=203$	59	11	1
$\Sigma u_3=Q+\ 357$	$u_3=262$	70	12	1
$\Sigma u_4=Q+\ 619$	$u_4=332$	82	13	1
$\Sigma u_5=Q+\ 951$	$u_5=414$	95	14	
$\Sigma u_6=Q+1365$	$u_6=509$	109		
	$u_7=618$			
	&c.			

Expressions, corresponding to those found in Examples (XI.) and (XII.), from equidifferent terms of a series, may, in a similar manner, be found from the sums of a series; as in the above example, the consecutive terms of the series were found from the sums in a similar manner to that by which they were found from the equidifferent terms of a series in Example (X.).

(XV.) Find the analytical expression for u_v in a series of ascending powers of v, when $S_5u_1 = 155$, $S_{10}u_1 = 2{,}410$, $S_{15}u_1 = 12{,}265$, and $S_{20}u_1 = 38{,}970$, Δ^3 being constant.

Since Δ^3 is constant, the highest power of the variable in the analytical expression is the third, so that we obtain from formula No. 6 and the values given above the following four linear equations, and from them we find the values of the constants in the analytical expression for u_v by continuous subtraction, thus,

$$\begin{array}{rrrrrr}
S_5\,u_1 = & 5c_0 + & 15c_1 + & 55c_2 + & 225c_3 = & 155 \\
S_{10}u_1 = & 10\,,,+ & 55\,,,+ & 385\,,,+ & 3{,}025\,,,= & 2{,}410 \\
S_{15}u_1 = & 15\,,,+ & 120\,,,+ & 1{,}240\,,,+ & 14{,}400\,,,= & 12{,}265 \\
S_{20}u_1 = & 20\,,,+ & 210\,,,+ & 2{,}870\,,,+ & 44{,}100\,,,= & 38{,}970 \\
\hline
\end{array}$$

$$\begin{array}{rrrrrl}
5c_0 + & 15c_1 + & 55c_2 + & 225c_3 = & 155 & \quad (1) \\
5\,,,+ & 40\,,,+ & 330\,,,+ & 2{,}800\,,,= & 2{,}255 & \\
5\,,,+ & 65\,,,+ & 855\,,,+ & 11{,}375\,,,= & 9{,}855 & \\
5\,,,+ & 90\,,,+ & 1{,}630\,,,+ & 29{,}700\,,,= & 26{,}705 & \\
\hline
\end{array}$$

$$\begin{array}{rrrrl}
25c_1 + & 275c_2 + & 2{,}575c_3 = & 2{,}100 & \quad (2) \\
25\,,,+ & 525\,,,+ & 8{,}575\,,,= & 7{,}600 & \\
25\,,,+ & 775\,,,+ & 18{,}325\,,,= & 16{,}850 & \\
\hline
\end{array}$$

$$\begin{array}{rrrl}
250c_2 + & 6{,}000c_3 = & 5{,}500 & \quad (3) \\
250\,,,+ & 9{,}750\,,,= & 9{,}250 & \\
\hline
\end{array}$$

$$\begin{array}{rrl}
3{,}750c_3 = & 3{,}750 & \quad (4) \\
\hline
\end{array}$$

We have from equation

(4) $3{,}750c_3 = 3{,}750$ and $c_3 = +1$

(3) $250c_2 = 5{,}500 - 6{,}000c_3 = 5{,}500 - 6{,}000 = -500$ $c_2 = -2$

(2) $25c_1 = 2{,}100 - 2{,}575c_3 - 275c_2 = 2{,}100 - 2{,}575 + 550 = 75$ $c_1 = +3$

(1) $5c_0 = 155 - 225c_3 - 55c_2 - 15c_1 = 155 - 225 + 110 - 45 = -5$ $c_0 = -1$

Therefore $u_v = -1 + 3v^1 - 2v^2 + v^3$.

The foregoing formulæ and examples may also be applied to the solution of problems in series of which the successive orders of differences always decrease, and after a finite number of orders the succeeding ones become so small that they may be neglected without affecting the result,—for example, such series as those for the Logarithms or Roots of consecutive numbers.

FIGURATE NUMBERS.

As the orders of Figurate Numbers are of frequent occurrence, there are given in Diagram III. the values of the first 8 orders to 20 terms. The first order is composed of a series of units; and the terms of each succeeding order are equal to the sums up to and including that term of the preceding order, so that the nth term of the pth order is equal to the sum of n terms of the $(p-1)$th order.

The following are the formulæ for the nth term, and also for the sum of n terms of the different orders,

The nth term of the

1st order $= n^0$

2nd „ $= n^1$

3rd „ $= \frac{n(n+1)}{1.2}$

4th „ $= \frac{n(n+1)(n+2)}{1.2.3}$

5th „ $= \frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$

6th „ $= \frac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}$

pth „ $= \frac{n(n+1)(n+2)\ldots\ldots(n+p-2)}{1.2.3\ldots\ldots(p-1)}$.

The sum of n terms of the

1st order $= n^1$

2nd „ $= \frac{n(n+1)}{1.2}$

3rd „ $= \frac{n(n+1)(n+2)}{1.2.3}$

4th „ $= \frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$

5th ,, $= \dfrac{n(n+1)(n+2)(n+3)(n+4)}{1.2.3.4.5}$

6th ,, $= \dfrac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{1.2.3.4.5.6}$

pth ,, $= \dfrac{n(n+1)(n+2)\ldots\ldots(n+p-1)}{1.2.3\ldots\ldots p}$.

The numbers in the Diagram read diagonally in the direction of the ruled lines, and connected by the positive sign are the coefficients of the expansions of a binomial to positive powers, while the terms of the several orders themselves, with their signs alternately positive and negative, are the coefficients of similar expansions to negative powers, thus the coefficients of the expansion of

$(a+x)^1$ are $1+1$
$(a+x)^2$,, $1+2+1$
$(a+x)^3$,, $1+3+3+1$
&c. &c.

and of

$(a+x)^{-1}$ are $1-1+1-1+$ &c.
$(a+x)^{-2}$,, $1-2+3-4+$ &c.
$(a+x)^{-3}$,, $1-3+6-10+$ &c.
&c. &c.

FORMULÆ.

FINITE DIFFERENCES.

1. $$u_v = c_0 + c_1 v^1 + c_2 v^2 + c_3 v^3 + c_4 v^4 + \&c.$$

2(*a*). $$u_{x+n} = u_x + \frac{n}{1}\Delta^1 u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \frac{n(n-1)(n-2)}{1.2.3}\Delta^3 u_x + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\Delta^4 u_x + \&c.$$

(*b*). $$u_{x-n} = u_x - \frac{n}{1}\Delta^1 u_x + \frac{n(n+1)}{1.2}\Delta^2 u_x - \frac{n(n+1)(n+2)}{1.2.3}\Delta^3 u_x + \frac{n(n+1)(n+2)(n+3)}{1.2.3.4}\Delta^4 u_x + \&c.$$

(*c*). $$u_{0(b)+\frac{a}{b}(b)} = u_{0(b)} + a\frac{\delta^1 u_{0(b)}}{1.b} + (a^2 - ab)\frac{\delta^2 u_{0(b)}}{1.2.b^2} + (a^3 - 3a^2b + 2ab^2)\frac{\delta^3 u_{0(b)}}{1.2.3.b^3} + (a^4 - 6a^3b + 11a^2b^2 - 6ab^3)\frac{\delta^4 u_{0(b)}}{1.2.3.4.b^4} + \&c.$$

3. $$\Delta^n u_x = u_{x+n} - \frac{n}{1}u_{x+n-1} + \frac{n(n-1)}{1.2}u_{x+n-2} - \frac{n(n-1)(n-2)}{1.2.3}u_{x+n-3} + \ldots \ldots \mp \frac{n}{1}u_{x+1} \pm u_x.$$

4. $$\Sigma u_{x+n} = \Sigma u_x + \frac{n}{1}u_x + \frac{n(n-1)}{1.2}\Delta^1 u_x + \frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_x + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\Delta^3 u_x + \&c.$$

5. $$S_n u_x = (\Sigma u_{x+n} - \Sigma u_x) = \frac{n}{1}u_x + \frac{n(n-1)}{1.2}\Delta^1 u_x + \frac{n(n-1)(n-2)}{1.2.3}\Delta^2 u_x + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\Delta^3 u_x + \&c.$$

6. $$S_n u_x = n.c_0 + \frac{1}{2}n(n+1).c_1 + \frac{1}{6}n(n+1)(2n+1).c_2 + \frac{1}{4}n^2(n+1)^2.c_3 + \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1).c_4 + \frac{1}{12}n^2(n+1)^2(2n^2+2n-1).c_5 + \&c.$$

7. $$u_v = \frac{(v-b)(v-c)(v-d)\ \&c.}{(a-b)(a-c)(a-d)\ \&c.}\cdot u_a + \frac{(v-a)(v-c)(v-d)\ \&c.}{(b-a)(b-c)(b-d)\ \&c.}\cdot u_b + \frac{(v-a)(v-b)(v-d)\ \&c.}{(c-a)(c-b)(c-d)\ \&c.}\cdot u_c + \frac{(v-a)(v-b)(v-c)\ \&c.}{(d-a)(d-b)(d-c)\ \&c.}\cdot u_d + \&c.$$

FIGURATE NUMBERS.

The nth term of the pth order $= \dfrac{n(n+1)(n+2)\ldots\ldots(n+p-2)}{1.2.3\qquad\qquad\qquad(p-1)}$

The sum of n terms of the pth order $= \dfrac{n(n+1)(n+2)\ldots\ldots(n+p-2)(n+p-1)}{1.2.3\qquad\qquad\qquad\qquad(p-1).p}$.

BINOMIAL THEOREM.

$$(a+x)^n = a^n + \frac{n}{1}a^{n-1}x^1 + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \text{\&c.}$$

whether n be integral or fractional, positive or negative.

DIAGRAM I., *showing the Formation of the*

u	$\Delta^1 u$	$\Delta^2 u$	$\Delta^3 u$
u_x	$\Delta^1 u_x = u_{x+1} - u_x$	$\Delta^2 u_x = u_{x+2} - 2u_{x+1} + u_x$	$\Delta^3 u_x = u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x$
u_{x+1}	$\Delta^1 u_{x+1} = u_{x+2} - u_{x+1}$	$\Delta^2 u_{x+1} = u_{x+3} - 2u_{x+2} + u_{x+1}$	$\Delta^3 u_{x+1} = u_{x+4} - 3u_{x+3} + 3u_{x+2} - u_{x+1}$
u_{x+2}	$\Delta^1 u_{x+2} = u_{x+3} - u_{x+2}$	$\Delta^2 u_{x+2} = u_{x+4} - 2u_{x+3} + u_{x+2}$	$\Delta^3 u_{x+2} = u_{x+5} - 3u_{x+4} + 3u_{x+3} - u_{x+2}$
u_{x+3}	$\Delta^1 u_{x+3} = u_{x+4} - u_{x+3}$	$\Delta^2 u_{x+3} = u_{x+5} - 2u_{x+4} + u_{x+3}$	$\Delta^3 u_{x+3} = u_{x+6} - 3u_{x+5} + 3u_{x+4} - u_{x+3}$
u_{x+4}	$\Delta^1 u_{x+4} = u_{x+5} - u_{x+4}$	$\Delta^2 u_{x+4} = u_{x+6} - 2u_{x+5} + u_{x+4}$	$\Delta^3 u_{x+4} = u_{x+7} - 3u_{x+6} + 3u_{x+5} - u_{x+4}$
u_{x+5}	$\Delta^1 u_{x+5} = u_{x+6} - u_{x+5}$	$\Delta^2 u_{x+5} = u_{x+7} - 2u_{x+6} + u_{x+5}$	
u_{x+6}	$\Delta^1 u_{x+6} = u_{x+7} - u_{x+6}$		
u_{x+7}			

DIAGRAM II., *showing the Formation of the Sums and of the Terms of a Series when the Fourth Order of*

Σu		u	
Σu_{x-4}	$= \Sigma u_x - 4u_x + 10\Delta^1 u_x - 20\Delta^2 u_x + 35\Delta^3 u_x - 56c$	u_{x-4}	$= u_x - 4\Delta^1 u_x + 10\Delta^2 u_x - 20\Delta^3 u_x + 35c$
Σu_{x-3}	$= \Sigma u_x - 3$ „ $+ 6$ „ $- 10$ „ $+ 15$ „ $- 21c$	u_{x-3}	$= u_x - 3$ „ $+ 6$ „ $- 10$ „ $+ 15c$
Σu_{x-2}	$= \Sigma u_x - 2$ „ $+ 3$ „ $- 4$ „ $+ 5$ „ $- 6c$	u_{x-2}	$= u_x - 2$ „ $+ 3$ „ $- 4$ „ $+ 5c$
Σu_{x-1}	$= \Sigma u_x - 1$ „ $+ 1$ „ $- 1$ „ $+ 1$ „ $- 1c$	u_{x-1}	$= u_x - 1$ „ $+ 1$ „ $- 1$ „ $+ 1c$
Σu_x	$= \Sigma u_x$ which denotes the sum of the infinite number of terms backwards from u_{x-1}, and is an infinite constant quantity.	u_x	$= u_x$
Σu_{x+1}	$= \Sigma u_x + 1u_x$	u_{x+1}	$= u_x + 1\Delta^1 u_x$
Σu_{x+2}	$= \Sigma u_x + 2$ „ $+ 1\Delta^1 u_x$	u_{x+2}	$= u_x + 2$ „ $+ 1\Delta^2 u_x$
Σu_{x+3}	$= \Sigma u_x + 3$ „ $+ 3$ „ $+ 1\Delta^2 u_x$	u_{x+3}	$= u_x + 3$ „ $+ 3$ „ $+ 1\Delta^3 u_x$
Σu_{x+4}	$= \Sigma u_x + 4$ „ $+ 6$ „ $+ 4$ „ $+ 1\Delta^3 u_x$	u_{x+4}	$= u_x + 4$ „ $+ 6$ „ $+ 4$ „ $+ 1c$
Σu_{x+5}	$= \Sigma u_x + 5$ „ $+ 10$ „ $+ 10$ „ $+ 5$ „ $+ 1c$	u_{x+5}	$= u_x + 5$ „ $+ 10$ „ $+ 10$ „ $+ 5c$
Σu_{x+6}	$= \Sigma u_x + 6$ „ $+ 15$ „ $+ 20$ „ $+ 15$ „ $+ 6c$	u_{x+6}	$= u_x + 6$ „ $+ 15$ „ $+ 20$ „ $+ 15c$
Σu_{x+7}	$= \Sigma u_x + 7$ „ $+ 21$ „ $+ 35$ „ $+ 35$ „ $+ 21c$	u_{x+7}	$= u_x + 7$ „ $+ 21$ „ $+ 35$ „ $+ 35c$

Differences from the Terms of a Series.

$\Delta^4 u$	
$\Delta^4u_x \quad = u_{x+4} - 4u_{x+3} + 6u_{x+2} - 4u_{x+1} + u_x$	And so on.
$\Delta^4u_{x+1} \quad = u_{x+5} - 4u_{x+4} + 6u_{x+3} - 4u_{x+2} + u_{x+1}$	
$\Delta^4u_{x+2} \quad = u_{x+6} - 4u_{x+5} + 6u_{x+4} - 4u_{x+3} + u_{x+2}$	
$\Delta^4u_{x+3} \quad = u_{x+7} - 4u_{x+6} + 6u_{x+5} - 4u_{x+4} + u_{x+3}$	

---

from a given Term of the Series, u_x, and its successive Orders of Differences, Differences is constant.

$\Delta^1 u$		$\Delta^2 u$		$\Delta^3 u$		$\Delta^4 u$	
Δ^1u_{x-4}	$=\Delta^1u_x - 4\Delta^2u_x + 10\Delta^3u_x - 20c$	Δ^2u_{x-4}	$=\Delta^2u_x - 4\Delta^3u_x + 10c$	Δ^3u_{x-4}	$=\Delta^3u_x - 4c$	Δ^4u_{x-4}	$=$
Δ^1u_{x-3}	$=\Delta^1u_x - 3$ „ $+ 6$ „ $-10c$	Δ^2u_{x-3}	$=\Delta^2u_x - 3$ „ $+ 6c$	Δ^3u_{x-3}	$=\Delta^3u_x - 3c$	Δ^4u_{x-3}	$=$
Δ^1u_{x-2}	$=\Delta^1u_x - 2$ „ $+ 3$ „ $- 4c$	Δ^2u_{x-2}	$=\Delta^2u_x - 2$ „ $+ 3c$	Δ^3u_{x-2}	$=\Delta^3u_x - 2c$	Δ^4u_{x-2}	$=$
Δ^1u_{x-1}	$=\Delta^1u_x - 1$ „ $+ 1$ „ $- 1c$	Δ^2u_{x-1}	$=\Delta^2u_x - 1$ „ $+ 1c$	Δ^3u_{x-1}	$=\Delta^3u_x - 1c$	Δ^4u_{x-1}	$=$
Δ^1u_x	$=\Delta^1u_x$	Δ^2u_x	$=\Delta^2u_x$	Δ^3u_x	$=\Delta^3u_x$	Δ^4u_x	$=$
Δ^1u_{x+1}	$=\Delta^1u_x + 1\Delta^2u_x$	Δ^2u_{x+1}	$=\Delta^2u_x + 1\Delta^3u_x$	Δ^3u_{x+1}	$=\Delta^3u_x + 1c$	Δ^4u_{x+1}	$=$
Δ^1u_{x+2}	$=\Delta^1u_x + 2$ „ $+ 1\Delta^3u_x$	Δ^2u_{x+2}	$=\Delta^2u_x + 2$ „ $+ 1c$	Δ^3u_{x+2}	$=\Delta^3u_x + 2c$	Δ^4u_{x+2}	$=$
Δ^1u_{x+3}	$=\Delta^1u_x + 3$ „ $+ 3$ „ $+ 1c$	Δ^2u_{x+3}	$=\Delta^2u_x + 3$ „ $+ 3c$	Δ^3u_{x+3}	$=\Delta^3u_x + 3c$	Δ^4u_{x+3}	$=$
Δ^1u_{x+4}	$=\Delta^1u_x + 4$ „ $+ 6$ „ $+ 4c$	Δ^2u_{x+4}	$=\Delta^2u_x + 4$ „ $+ 6c$	Δ^3u_{x+4}	$=\Delta^3u_x + 4c$	Δ^4u_{x+4}	$=$
Δ^1u_{x+5}	$=\Delta^1u_x + 5$ „ $+10$ „ $+10c$	Δ^2u_{x+5}	$=\Delta^2u_x + 5$ „ $+10c$	Δ^3u_{x+5}	$=\Delta^3u_x + 5c$	Δ^4u_{x+5}	$=$
Δ^1u_{x+6}	$=\Delta^1u_x + 6$ „ $+15$ „ $+20c$	Δ^2u_{x+6}	$=\Delta^2u_x + 6$ „ $+15c$	Δ^3u_{x+6}	$=\Delta^3u_x + 6c$	Δ^4u_{x+6}	$=$
Δ^1u_{x+7}	$=\Delta^1u_x + 7$ „ $+21$ „ $+35c$	Δ^2u_{x+7}	$=\Delta^2u_x + 7$ „ $+21c$	Δ^3u_{x+7}	$=\Delta^3u_x + 7c$	Δ^4u_{x+7}	$=$

Diagram III., *giving the first* 20 *Terms of the first* 8 *Orders of Figurate Numbers.*

Term.	I. Order.	II. Order.	III. Order.	IV. Order.	V. Order.	VI. Order.	VII. Order.	VIII. Order.
1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8
3	1	3	6	10	15	21	28	36
4	1	4	10	20	35	56	84	120
5	1	5	15	35	70	126	210	330
6	1	6	21	56	126	252	462	792
7	1	7	28	84	210	462	924	1,716
8	1	8	36	120	330	792	1,716	3,432
9	1	9	45	165	495	1,287	3,003	6,435
10	1	10	55	220	715	2,002	5,005	11,440
11	1	11	66	286	1,001	3,003	8,008	19,448
12	1	12	78	364	1,365	4,368	12,376	31,824
13	1	13	91	455	1,820	6,188	18,564	50,388
14	1	14	105	560	2,380	8,568	27,132	77,520
15	1	15	120	680	3,060	11,628	38,760	116,280
16	1	16	136	816	3,876	15,504	54,264	170,544
17	1	17	153	969	4,845	20,349	74,613	245,157
18	1	18	171	1,140	5,985	26,334	100,947	346,104
19	1	19	190	1,330	7,315	33,649	134,596	480,700
20	1	20	210	1,540	8,855	42,504	177,100	657,800

Note.—Most of the Examples given may be solved by different methods from those adopted in the context; thus, instead of using fractions in the solution of Nos. X., XI., and XIV., the inverse method may be employed; for instance, No. XI. may be solved as follows:

By formula No. 2(a).

$$\begin{array}{llllllll}
u_{0(1)+0b}= & 1u_{0(1)} & +0\,.\dfrac{\Delta^1u_{0(1)}}{1} & +\;0 & .\dfrac{\Delta^2u_{0(1)}}{1.2} & +\;0 & .\dfrac{\Delta^3u_{0(1)}}{1.2.3} & =u_{0(b)} \\
u_{0(1)+1b}= & 1\;\text{,,} & +1b.\;\text{,,} & +b(b-1). & \text{,,} & +b(b-1)(b-2). & \text{,,} & =u_{1(b)} \\
u_{0(1)+2b}= & 1\;\text{,,} & +2b.\;\text{,,} & +2b(2b-1). & \text{,,} & +2b(2b-1)(2b-2). & \text{,,} & =u_{2(b)} \\
u_{0(1)+3b}= & 1\;\text{,,} & +3b.\;\text{,,} & +3b(3b-1). & \text{,,} & +3b(3b-1)(3b-2). & \text{,,} & =u_{3(b)} \\
\hline
 & 1u_{0(1)} & +0\,.\dfrac{\Delta^1u_{0(1)}}{1} & +\;0 & .\dfrac{\Delta^2u_{0(1)}}{1.2} & +\;0 & .\dfrac{\Delta^3u_{0(1)}}{1.2.3} & =u_{0(b)} \\
 & 1\;\text{,,} & +1b.\;\text{,,} & +(b^2-1b). & \text{,,} & +(b^3-3b^2+2b). & \text{,,} & =u_{1(b)} \\
 & 1\;\text{,,} & +2b.\;\text{,,} & +(4b^2-2b). & \text{,,} & +(8b^3-12b^2+4b). & \text{,,} & =u_{2(b)} \\
 & 1\;\text{,,} & +3b.\;\text{,,} & +(9b^2-3b). & \text{,,} & +(27b^3-27b^2+6b). & \text{,,} & =u_{3(b)}
\end{array}$$

$$\begin{array}{llllllll}
1b\cdot\dfrac{\Delta^1u_{0(1)}}{1} & +(b^2-b). & \dfrac{\Delta^2u_{0(1)}}{1.2} & +(b^3-3b^2+2b). & \dfrac{\Delta^3u_{0(1)}}{1.2.3} & =\Delta^1u_{0(b)} & (1) \\
1b.\;\text{,,} & +(3b^2-b). & \text{,,} & +(7b^3-9b^2+2b). & \text{,,} & =\Delta^1u_{1(b)} & \\
1b.\;\text{,,} & +(5b^2-b). & \text{,,} & +(19b^3-15b^2+2b). & \text{,,} & =\Delta^1u_{2(b)} & \\
\hline
 & (2b^2). & \dfrac{\Delta^2u_{0(1)}}{1.2} & +(6b^3-6b^2). & \dfrac{\Delta^3u_{0(1)}}{1.2.3} & =\Delta^2u_{0(b)} & (2) \\
 & (2b^2). & \text{,,} & +(12b^3-6b^2). & \text{,,} & =\Delta^2u_{1(b)} & \\
\hline
 & & & (6b^3). & \dfrac{\Delta^3u_{0(1)}}{1.2.3} & =\Delta^3u_{0(b)} & (3) \\
\hline
\end{array}$$

We obtain from equation

$$(3)\qquad 6b^3\cdot\frac{\Delta^3u_{0(1)}}{1.2.3}=\Delta^3u_{0(b)}$$

$$\Delta^3u_{0(1)}=\frac{\Delta^3u_{0(b)}}{b^3}$$

$$(2)\qquad 2b^2\cdot\frac{\Delta^2u_{0(1)}}{1.2}=\Delta^2u_{0(b)}-(6b^3-6b^2)\cdot\frac{\Delta^3u_{0(1)}}{1.2.3}$$

$$=\Delta^2u_{0(b)}-(b^3-b^2)\,.\,\frac{\Delta^3u_{0(b)}}{b^3}$$

(By substituting the value of $\Delta^3u_{0(1)}$ as found above.)

$$\Delta^2u_{0(1)}=\frac{\Delta^2u_{0(b)}}{b^2}-(b-1)\cdot\frac{\Delta^3u_{0(b)}}{b^3}$$

$$(1)\qquad 1b\cdot\frac{\Delta^1u_{0(1)}}{1}=\Delta^1u_{0(b)}-(b^2-b)\cdot\frac{\Delta^2u_{0(1)}}{1.2}-(b^3-3b^2+2b)\cdot\frac{\Delta^3u_{0(1)}}{1.2.3}$$

$$=\Delta^1u_{0(b)}-\left(\frac{b^2-b}{1.2}\cdot\frac{\Delta^2u_{0(b)}}{b^2}-\frac{3b^3-6b^2+3b}{1.2.3}\cdot\frac{\Delta^3u_{0(b)}}{b^3}\right)-\frac{b^3-3b^2+2b}{1.2.3}\cdot\frac{\Delta^3u_{0(}}{b^3}$$

$$=\Delta^1u_{0(b)}-\frac{b^2-b}{1.2}\cdot\frac{\Delta^2u_{0(b)}}{b^2}+\frac{2b^3-3b^2+1b}{1.2.3}\cdot\frac{\Delta^3u_{0(b)}}{b^3}$$

$$\Delta^1u_{0(1)}=\frac{\Delta^1u_{0(b)}}{b}-\frac{b-1}{1.2}\cdot\frac{\Delta^2u_{0(b)}}{b^2}+\frac{(b-1)(2b-1)}{1.2.3}\cdot\frac{\Delta^3u_{0(b)}}{b^3}$$

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